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A lower bound on the number of removable ears of 1-extendable graphs[☆]

Shaohui Zhai [a], Cláudio L. Lucchesi [b], Xiaofeng Guo [c,*]

[a] *Department of Mathematics and Physics, Xiamen University of Technology, Xiamen Fujian 361024, China*

[b] *Institute of Computing, UNICAMP, Caixa Postal 6176, 13084-971, Campinas, SP, Brazil*

[c] *School of Mathematical Sciences, Xiamen University, Xiamen Fujian 361005, China*

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ABSTRACT

Let G be a 1-extendable graph distinct from K_2 and C_{2n}. A classical result of Lovász and Plummer (1986) [5, Theorem 5.4.6] states that G has a removable ear. Carvalho et al. (1999) [3] proved that G has at least $\Delta(G)$ edge-disjoint removable ears, where $\Delta(G)$ denotes the maximum degree of G. In this paper, the authors improve the lower bound and prove that G has at least $m(G)$ edge-disjoint removable ears, where $m(G)$ denotes the minimum number of perfect matchings needed to cover all edges of G.

1. Introduction

All graphs considered in this paper are connected and may have multiple edges but no loops. For terminology and notation not defined in this paper, the reader is referred to [3,5].

An edge e in a graph G is *admissible* if e appears in some perfect matching in G. Let n be a positive integer. A graph G is *n-extendable* if it is connected, contains a matching of size n, and every matching of size n can extend to a perfect matching of G. Plummer introduced the concept and proved the following result:

Theorem 1.1 (*Plummer [6]*). *Let n be a positive integer. If G is an n-extendable graph with $2n \leq |V(G)| - 2$, then G is $(n+1)$-connected.*

Let G be a graph. For any subset $S \subset V(G)$ and $\overline{S} = V(G) \setminus S$, let $\partial(S)$ (resp. $\partial(\overline{S})$) denote the set of edges of G with exactly one end vertex in S (resp. $\overline{S}$); this is the *edge cut* generated by S (also by $\overline{S}$). When C is an edge cut generated by S (also by $\overline{S}$), the graph obtained from G by contracting S (resp. $\overline{S}$) to a single vertex s (resp. $\overline{s}$) is said to be a *C-contraction* of G, and is denoted by G/S (resp. $G/\overline{S}$). An edge cut generated by S (also by $\overline{S}$) is *trivial* if one of S and $\overline{S}$ is a singleton.

Let G be a 1-extendable graph. An edge cut C of G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. Clearly, any trivial cut is a tight cut of G. The proof of the following result is immediate:

Lemma 1.2. *If G is a 1-extendable graph and C is a tight cut of G, then both the C-contractions of G are 1-extendable.*

For a 1-extendable graph G, let e and f be any two edges of G. We say that *e depends on f* if every perfect matching that contains e also contains f. Let $e \Rightarrow f$ indicate that e depends on f. Clearly, $\Rightarrow$ is reflexive and transitive. Two edges e and f are *mutually dependent* (simply, $e \Leftrightarrow f$) if $e \Rightarrow f$ and $f \Rightarrow e$. Obviously, $\Leftrightarrow$ is an equivalence relation on $E(G)$. It is convenient

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* Corresponding author.
E-mail address: xfguo@xmu.edu.cn (X. Guo).

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to visualize $\Rightarrow$ in terms of the digraph it defines on the equivalence classes of $E(G)$. Carvalho [2] introduced the following definition. Let $D(G)$ be the digraph, called the *dependence relation graph of* G, whose vertex set consists of all equivalence classes on $E(G)$ and between two vertices X and Y of $D(G)$ (two equivalence classes of $E(G)$) there is an arc (X, Y) if and only if any perfect matching of G that contains all edges in X also contains all edges in Y. Clearly, $D(G)$ is acyclic. The sources in this digraph are called *minimal classes*. For an edge e of G, let $[e]$ denote equivalence class of $E(G)$ containing e. Given an edge e of G, consider the subdigraph of $D(G)$ induced by the set of all the equivalence classes $[f]$ such that $f \Rightarrow e$. A minimal class in this subdigraph of $D(G)$ is clearly a minimal class in $D(G)$, which is said to be *induced by* e. Carvalho, Lucchesi, and Murty proved the following two fundamental results:

Theorem 1.3 (*Carvalho, Lucchesi, Murty [3]*). *If G is a 1-extendable graph and Q is a minimal class of G, then any edge not in Q is admissible in $G - Q$. In particular, if $G - Q$ is connected, then $G - Q$ is 1-extendable.*

A graph G is *bicritical* if the deletion of any two vertices of G results in a graph with a perfect matching. In particular, a bicritical graph G is said to be a *brick* if it is 3-connected.

Theorem 1.4 (*Carvalho, Lucchesi, Murty [3]*). *If G is a brick, and Q is an equivalence class of $E(G)$, then $|Q| \leq 2$. Moreover, if $Q = 2$, then $G - Q$ is bipartite.*

Let G be a 1-extendable graph. A *single ear* of G is a path of odd length whose internal vertices have degree two in G. A *double ear* is a pair of vertex-disjoint single ears. *An ear* in G is a singer ear or double ear. Let R be a singer ear or a double ear of G. Denote by $G - R$ the graph obtained from G by deleting the edges and internal vertices of the constituent paths of R. A single ear R of G is *removable* if the graph $G - R$ is 1-extendable. A removable single ear of length one is called a *removable edge*. A removable double ear $R = (P_1, P_2)$ in G is similarly defined, where neither P_1 nor P_2 is a removable single ear in G. A removable double ear whose ears are paths of length one is called a *removable doubleton*. A *removable ear* in G is either a removable single ear or a removable double ear. For convenience, denote by $\rho(G)$ the number of removable ears of G.

Lovász and Plummer [5, Theorem 5.4.6] proved that every 1-extendable graph G distinct from K_2 has a removable ear. In fact, the proof of that Theorem readily implies the following result:

Theorem 1.5. *Let G be a 1-extendable graph, and let $R_1, R_2, \ldots, R_t (t \geq 2)$ be edge-disjoint single ears of G. If $G - R_1 - R_2 - \cdots - R_t$ is 1-extendable, then there are **at most** two single ears R_i and R_j such that $G - R_i - R_j$ is also 1-extendable.*

Lovász [4] proved that every brick different from K_4 and $\overline{C_6}$ has a removable edge. Later, Carvalho, Lucchesi, and Murty improved the lower bounds of the number of removable ears and removable edges and gave the following results:

Theorem 1.6 (*Carvalho, Lucchesi, Murty [3]*). *If G is a 1-extendable graph distinct from K_2 and C_{2n}, then $\rho(G) \geq \Delta(G)$.*

Theorem 1.7 (*Carvalho, Lucchesi, Murty [3]*). *Let G be a brick. If G is distinct from K_4 and $\overline{C_6}$, then it has at least $\Delta(G) - 2$ removable edges.*

Let G be a 1-extendable graph. The *excessive index* [1] of G, denoted $m(G)$, is the minimum number of perfect matchings needed to cover all edges of G. In particular, define $m(G) = \infty$ if G is not 1-extendable. Clearly, $m(G) \geq \Delta(G)$ for any graph G.

Theorem 1.8 (*Zhai and Guo [7]*). *If G is a 1-extendable **graph**, $D(G)$ is the dependence relation digraph of G, and $\mathcal{Q}$ is the collection of minimal classes in $D(G)$, then $m(G) \leq |\mathcal{Q}|$.*

In this paper, we improve the lower bound of Theorem 1.6 by proving that any 1-extendable graph G different from K_2 and C_{2n} has at least $m(G)$ edge-disjoint removable ears.

2. Lower bounds for the numbers of removable ears and removable edges

Let C be a tight cut generated by S, and let G_1 and G_2 be the two C-contractions of G, where G_1 is obtained by contracting S to s_1, and G_2 is obtained by contracting $\overline{S}$ to s_2.

Theorem 2.1. *For $i = 1, 2$, let $\mathcal{M}_i$ denote a collection of perfect matchings of G_i that covers the set of edges of G_i. There exists a collection $\mathcal{M}$ of perfect matchings of G such that $\mathcal{M}$ covers all edges of G and*

$$|\mathcal{M}| = |\mathcal{M}_1| + |\mathcal{M}_2| - |C|.$$

Proof. Let e be any edge of C. For $i = 1, 2$, let $\mathcal{M}_i(e)$ denote the subcollection of $\mathcal{M}_i$ consisting of those perfect matchings that contain edge e. Let $M_i(e)$ be an arbitrary element of $\mathcal{M}_i(e)$. Let

$$\mathcal{M}(e) = \{M \cup M_2(e) : M \in \mathcal{M}_1(e)\} \cup \{M \cup M_1(e) : M \in \mathcal{M}_2(e)\}.$$

Clearly, any element of $\mathcal{M}(e)$ is a perfect matching of G which contains e. In addition, we have $|\mathcal{M}(e)| = |\mathcal{M}_1(e)| + |\mathcal{M}_2(e)| - 1$ and $\bigcup_{e \in C} \mathcal{M}(e)$ can cover all edges of G. Let $\mathcal{M} = \bigcup_{e \in C} \mathcal{M}(e)$, then $|\mathcal{M}| = |\mathcal{M}_1| + |\mathcal{M}_2| - |C|$. □

Lemma 2.2. *For $i = 1, 2$, let R_i be a removable ear of G_i. If $E(R_1)$ and $E(R_2)$ intersect and if C has three or more edges, then $E(R_1) \cup E(R_2)$ spans a removable ear of G.*

Proof. Assume that $E(R_1)$ and $E(R_2)$ have a common edge e. Assume also that C has three or more edges. Clearly, edge e lies in C. Therefore, e is the only edge of R_i in C, for $i = 1, 2$. We deduce that $E(R_1) \cap E(R_2) = \{e\}$.

If $T = E(R_1) \cup E(R_2)$, then $G[T]$ is the union of a collection $\mathcal{P} = \{P_i : 1 \le i \le t\}$ of vertex-disjoint single ears of G, where $1 \le t \le 3$. Moreover, if $t = 3$, then R_1 and R_2 are both double ears. Let $H = G - \bigcup_{P_i \in \mathcal{P}} P_i$. We assert that H is 1-extendable. We first prove that H is connected. If $G_1 - R_1$ contains only two vertices, then $G_1 - R_1 - s_1$ is the vertex graph. If $G_1 - R_1$ has four or more vertices, then, since it is 1-extendable, it is 2-connected, by Theorem 1.1. In both alternatives, $G_1 - R_1 - s_1$ is connected. That is, $(G - R_1)[\overline{S}]$ is connected. Likewise, $(G - R_2)[S]$ is connected. Moreover, $C - e$ is non-null. We deduce that H is connected. Let us now prove that every edge of H is admissible. Let f be any edge of H. Adjust notation so that f lies in $G_1 - R_1$. The graph $G_1 - R_1$ is 1-extendable; therefore, $G_1 - R_1$ has a perfect matching, M_1, that contains edge f. Let g denote the edge of M_1 in C. Edge g is distinct from e and therefore does not lie in T. Graph $G_2 - R_2$ is 1-extendable; thus it has a perfect matching, M_2, that contains edge g. Therefore, $M_1 \cup M_2$ is a perfect matching of H that contains edge f. We deduce that f is admissible in H. That conclusion holds for each edge f of H. Indeed, H is 1-extendable.

Let $\mathcal{P}'$ be a non-null proper subcollection of $\mathcal{P}$, and let G' denote the graph obtained from G by the removal of the single ears of $\mathcal{P}'$. We assert that G' is not 1-extendable. For this, assume the contrary. Let $C' = C \cap E(G')$, and, for $i = 1, 2$, let $G'_i = G_i[E(G') \cap E(G_i)]$. Clearly, G'_1 and G'_2 are the C'-contractions of G'. Every perfect matching of G' is a perfect matching of G. Therefore, C' is a tight cut of G'. By Lemma 1.2, G'_1 and G'_2 are both 1-extendable. This implies that either not all constituent single ears of R_1 were removed from G_1, or not all constituent single ears of R_2 were removed from G_2, yet a 1-extendable graph was obtained. This is a contradiction to the definition of removable (double) ear. As asserted, G' is not 1-extendable.

We deduce that if $t > 1$ then no single ear of $\mathcal{P}$ is removable. In particular, if $t = 2$ then the two single ears of $\mathcal{P}$ constitute a removable double ear. To complete the proof, it now remains to prove that $t \le 2$. For this, assume that $t = 3$. Then R_1 and R_2 are both double ears. By Theorem 1.5, $\mathcal{P}$ has a non-null proper collection of single ears whose removal from G yields a 1-extendable graph, a contradiction. We conclude that $t \le 2$. As asserted, T spans in G a removable ear. □

Theorem 2.3. *For $i = 1, 2$, let $\mathcal{R}_i$ denote a collection of edge-disjoint removable ears of G_i. If neither G_1 nor G_2 is a cycle, then there exists a collection $\mathcal{R}$ of edge-disjoint removable ears of G such that*

$$|\mathcal{R}| \ge |\mathcal{R}_1| + |\mathcal{R}_2| - |C|.$$

Proof. Consider first the case in which $|C| = 2$. In this case, let $\mathcal{R}$ be the collection of ears of $\mathcal{R}_1 \cup \mathcal{R}_2$ that do not contain any edge in C. Every ear in $\mathcal{R}$ is removable in G. Moreover, by hypothesis G_i is not C_{2n}. Thus, if G_i has a removable ear that contains an edge of C, then that ear contains both edges of C. We deduce that

$$|\mathcal{R}| \ge |\mathcal{R}_1| - 1 + |\mathcal{R}_2| - 1,$$

and the statement is true in this case.

We may thus assume that C has three or more edges. In that case, every ear of G_i, $i = 1, 2$, contains at most one edge in C. For $i = 1, 2$, let C_i denote the set of edges of C that lie in some ear of $\mathcal{R}_i$, let $\mathcal{R}'_i$ denote the set of ears of $\mathcal{R}_i$ that do not contain any edge in C. For each edge e in C_i, let $R_i(e)$ denote the ear of $\mathcal{R}_i$ that contains edge e. For each edge e in $C_1 \cap C_2$, let $R(e)$ be the subgraph of G spanned by $E(R_1(e)) \cup E(R_2(e))$. By Lemma 2.2, $R(e)$ is a removable ear of G. Now,

$$\mathcal{R} = \mathcal{R}'_1 \cup \mathcal{R}'_2 \cup \{R(e) : e \in C_1 \cap C_2\}$$

is a collection of edge-disjoint removable ears of G. Moreover,

$$|\mathcal{R}| = |\mathcal{R}_1| - |C_1| + |\mathcal{R}_2| - |C_2| + |C_1 \cap C_2| \ge |\mathcal{R}_1| + |\mathcal{R}_2| - |C|.$$

In all alternatives, the asserted inequality holds. □

3. The main result

Theorem 3.1. *If G is a brick, then $\rho(G) \ge m(G)$.*

Proof. Let $D(G)$ be the dependence relation digraph of G, and let $\mathcal{Q}$ be the set of minimal classes in $D(G)$. Clearly, any two minimal classes of $\mathcal{Q}$ are disjoint from each other. Let Q be any class in $\mathcal{Q}$. By Theorem 1.4, $|Q| \le 2$. Since G is 3-connected, $G - Q$ is connected. By Theorem 1.3, $G - Q$ is 1-extendable. This conclusion holds for each Q in $\mathcal{Q}$. Thus, G has at least $|\mathcal{Q}|$ edge-disjoint removable ears. By Theorem 1.8, G has at least $m(G)$ edge-disjoint removable ears. □

A bipartite 1-extendable graph G with bipartition (A, B) is called a *brace* if, for any two vertices u_1 and u_2 of A and any two vertices v_1 and v_2 of B, the graph $G - \{u_1, u_2, v_1, v_2\}$ has a perfect matching. The following result is well known.

Lemma 3.2 (*Carvalho, Lucchesi, Murty [3, Lemma 3.2]*). *If G is a brace on six or more vertices, then every edge of G is removable.*

Theorem 3.3. *If G is a brace distinct from K_2 and C_{2n}, then $\rho(G) \ge m(G)$.*

Proof. By induction on $|E(G)|$. If G has six or more vertices, then every edge of G is removable, by Lemma 3.2. If G has only two vertices, then it has multiple edges (since $G \ne K_2$), and again every edge is removable. In both alternatives, the asserted inequality holds trivially. We may thus assume that G is C_4, up to multiple edges. As G is not C_{2n}, G has multiple edges. Let u and v be two vertices of G joined by two or more edges, and let T denote the set of multiple edges of G that join u and v. Let

e be any edge of T, and let G' denote the graph $G-(T-e)$. If G' is C_4, then $\rho(G)=k+1=m(G)$ and the assertion holds, where $k=|T|$. We may thus assume that G' is not C_4. Now, the induction hypothesis applies to G'. Thus,

$$\rho(G)\geq\rho(G')-1+k\geq m(G')+(k-1)\geq m(G),$$

where the first inequality follows from the fact that every removable ear of G' that does not contain edge e is removable in G, the second inequality follows from the induction hypothesis, and the last inequality is immediate. □

Theorem 3.4. *If G is a 1-extendable graph distinct from K_2 and C_{2n}, then $\rho(G)\geq m(G)$.*

Proof. By induction on $|V(G)|$. If G is a brick then the assertion holds, by Theorem 3.1. If G is a brace then the assertion holds, by Theorem 3.3.

We may thus assume that G has a non-trivial tight cut, C. Let G_1 and G_2 denote the two C-contractions of G. If neither G_1 nor G_2 is C_{2n}, then

$$\rho(G)\geq\rho(G_1)+\rho(G_2)-|C|\geq m(G_1)+m(G_2)-|C|\geq m(G),$$

where the first inequality follows from Theorem 2.3, the second inequality follows from the induction hypothesis, and the third inequality follows from Theorem 2.1.

We may thus assume that one of G_1 and G_2 is C_{2n}. By hypothesis, G is not C_{2n}. Therefore, precisely one of G_1 and G_2 is C_{2n}. Adjust notation so that G_2 is a cycle. Then $|C|=2$. We assert that

$$\rho(G)\geq\rho(G_1). \tag{1}$$

Every removable ear of G_1 that does not contain any edge in C is removable in G. If no removable ear of G_1 contains edges in C, then (1) holds trivially. Thus we may assume that G_1 has a removable ear R_1 that contains edges in C. As G_1 is not a cycle, R_1 contains both edges in C. Let R be the subgraph of G spanned by $E(R_1)\cup E(G_2)$. Clearly, R is a removable ear of G, because $G-R=G_1-R_1$. Thus, (1) holds also in this case. Graph G_1 is not C_{2n} and clearly is not K_2. We deduce that

$$\rho(G)\geq\rho(G_1)\geq m(G_1)\geq m(G),$$

where the first inequality is given by (1), the second inequality follows from the induction hypothesis, and the last inequality follows from Theorem 2.1, as $m(G_2)=2=|C|$. □

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